182. Proposed by O. L. CALLICOTT, Gettysburg, S. Dak.

Find the value of $\sqrt{2}\,\sqrt[3]{2}\,\sqrt[4]{2}\,\sqrt[5]{2}\ldots\sqrt[1000]{2}$.

I. Solution by W. D. LAMBERT, Washington, D. C.

Let $\frac{1}{2}P$ be the required product.

Then $\log_{10}P=\log_{10}2(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\frac{1}{1000})$.

By the Euler-Bernoulli formula for reducing summation to integration,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{x}=v+\log_e x+\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}\cdots$$

v is Euler's constant$=0.57721, 56649\ldots$

$\therefore \log_{10}P=0.3010300(0.57721566+6.90775528+0.00050000-0.00000008\ldots)$ $=2.253351$.

$\therefore \frac{1}{2}P=89.603$, the required value.

II. Solution by S. A. COREY, Hiteman, Iowa.

If s is the value sought,

$$\log s=(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\tfrac{1}{1000})\log 2. \qquad (1)$$

If $\log(1+x)$ be developed by the formula given in Prize Problem 237, Calculus, we get

$$\log(1+x)=0+\frac{x}{2m}\left[\frac{1}{1+x}+1+2\left(\frac{1}{1+\frac{x}{m}}+\frac{1}{1+\frac{2x}{m}}+\ldots+\frac{1}{1+\frac{(m-1)x}{m}}\right)\right]$$

$$+\frac{B_1x^2}{m^2.2!}\left[\frac{1}{(1+x)^2}-1\right]-\frac{B_2x^4}{m^4.4!}\left[\frac{3!}{(1+x)^4}-3!\right]+\text{ etc.} \qquad (2)$$

If, now, x is an integer and m be taken equal to x, we have, as x approaches ∞,

$$\lim_{n\doteq\infty}\log n-\left[\tfrac{1}{2}+\sum_{r=2}^{r=\infty}r^{-1}\right]=-\left[\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}\ldots\right]=-b; \qquad (3)$$

whence if C be Euler's constant, .577,215,664,901,5..., $b=C-\frac{1}{2}$. Substituting in (2), transposing, adding $\frac{1}{1000}$ to each member, and reducing, we get, by making $(1+x)=1000$,

$$(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\tfrac{1}{1000})=\log 1000-1+\tfrac{1}{2000}+C-\left[\frac{B_1}{2.1000^2}-\frac{B_2}{4.1000^4}+\ldots\right]$$

$=6.485,470,860,55$, or $s=2^{6.485,470,860,55}=89.602,734.8\ldots$

Also solved by G. B. M. Zerr, J. Scheffer, and A. H. Holmes.